INDIAN STATISTICAL INSTITUTE

Probability Theory II: B. Math (Hons.) I
Semester II, Academic Year 2018-19
Backpaper Exam

Total Marks: 100 **Duration: 3 hours**

- **Please write your name on top of your answer-script.**
- **Show all your works and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. A random variable X is said to follow Laplace distribution with parameters $\mu \in \mathbb{R}$ and $\tau \in (0, \infty)$ (denoted by $X \sim \text{Laplace}(\mu, \tau)$) if X has a probability density function

$$f_X(x) = \frac{1}{2\tau} \exp\left(-\frac{|x-\mu|}{\tau}\right), \quad x \in \mathbb{R}.$$

   (a) (4 marks) Write down, with proper justification, an algorithm to simulate a random variable

$$Z \sim \text{Laplace}(0, 1).$$

   (b) (4 + 2 = 6 marks) If $Z \sim \text{Laplace}(0, 1)$, find a probability density function of $W := |Z|$. What distribution does W follow?

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2. Suppose X_1, X_2, X_3 is a random sample from exponential distribution with parameter $\lambda = 1$. Let $X_{(1)} < X_{(2)} < X_{(3)}$ be the order-statistics obtained from this sample.

   (a) (6 marks) Find a joint probability density function of the random vector
   $$\big(X_{(1)},\, X_{(2)} - X_{(1)},\, X_{(3)} - X_{(2)}\big).$$

   (b) (2 marks) Are $X_{(1)}$ and $X_{(3)} - X_{(1)}$ independent? Please justify your answer.

   (c) (2 marks) What distribution does $X_{(1)}$ follow?

3. A continuous random vector (X, Y) has a joint probability density function given by
   $$f_{X,Y}(x, y) = e^{-(x-y)} \quad \text{if } 0 < y < 1 \text{ and } x > y.$$

   (a) (4+4 = 8 marks) Find marginal probability density functions of X and Y.

   (b) (2 marks) Are X and Y independent? Please justify your answer.

   (c) (5+5 = 10 marks) Calculate the conditional probability density functions of X given Y and Y given X.

   (d) (5 marks) Compute $E(X|Y)$.

4. (5 marks) Suppose X follows a double-exponential distribution with a probability density function
   $$f_X(x) = \frac{1}{2} e^{-|x|}, \quad x \in \mathbb{R}.$$
   Compute the characteristic function of X.

5. (10 marks) Suppose $Y_1, Y_2, \ldots, Y_n$ is a random sample from a standard Cauchy distribution with common probability density function
   $$g(y) = \frac{1}{\pi(1 + y^2)}, \quad y \in \mathbb{R}.$$
   Compute a probability density function of $\frac{1}{n}\sum_{i=1}^{n} Y_i$. Justify all your steps.

6. State whether each of the following statements is true or false. If it is true, give a detailed proof. On the other hand, if it is false, produce a counter-example with full justification. If you both prove and disprove a statement, you will get a zero in that problem.

   (a) (10 marks) If $X_n \xrightarrow{d} X$, then $X_n^2 \xrightarrow{d} X^2$.

   (b) (10 marks) If X and Y are two independent random variables such that at least one of them has a continuous cumulative distribution function, then $P(X = Y) = 0$.

   (c) (10 marks) If X_1 and X_2 are i.i.d. random variables following exponential distribution with parameter $\lambda = 2$, then $V := \frac{4X_1+3X_2}{X_1+X_2}$ follows uniform distribution on the interval $(3, 4)$.

   (d) (10 marks) If (X, Y) is a bivariate normal random vector such that marginally both X and Y follow standard normal distribution, then $Corr(X^2, Y^2) = \big(Corr(X, Y)\big)^2$.